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Micu-type invariants for the Casimir operators of the Lie superalgebras osp(1, 2*n*): application to the algebra osp(1, 4)

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Abstract. A construction of the Casimir operators of higher degree of the Lie superalgebras osp(1, 2n) by a method which represents a generalisation of the method proposed by Micu for the construction of the Casimir operators of the Lie algebras, namely the algebras Sp(2n), is suggested.

The method applied has a close connection with Backhouse's method of construction of Casimir operators for the semisimple Lie superalgebras.

Above all the fourth-order coefficients of the Casimir operators of superalgebras osp(1, 2n) are shown to play a determining role for the Casimir operators of higher degrees.

1. Introduction

It is well known that in the theory of the classical semisimple Lie algebras (SSLAS) there exist some formally different methods, which enable us to construct the independent Casimir operators of higher degrees (Racah 1951, Okubo 1962, Perelomov and Popov 1965, Nwachuku and Rashid 1976, Agrawala 1979, Micu 1964). The formal differences of the particular methods of construction are usually connected with the specific choice of the basis in the corresponding Lie algebra.

The situation for the semisimple Lie superalgebras (SSGLS) is similar, e.g. three formally different methods have been suggested for algebras of the type osp(1, 2n), ($n = 1, 2, \ldots$), (Bednář and Šachl 1978a, b, 1979, Backhouse 1977, Jarvis and Green 1979) which are suitable for the construction of the Casimir operators of higher degrees. The particular Casimir operators of higher degrees are usually expressed in a global form relatively easily by means of the sums of specific products (of the corresponding degree) of generators of the given algebra.

However, for many specific applications, it is necessary to know the explicit formulae for the Casimir operators in terms of the independent generators.

To obtain these particular formulae, it is naturally desirable to apply the simplest possible method for their construction. Apart from the above mentioned methods (Bednář and Šachl 1978a, Backhouse 1977, Jarvis and Green 1979), a further method of construction of the independent higher-order Casimir operators for the superalgebras osp(1, 2n) exists—a generalisation of the well known Micu construction—which we present in this paper.

After introducing the general definition of the Backhouse type in § 2, the independent Casimir operators of the second and fourth degree of the superalgebra osp(1, 4) are studied in detail in § 3. Amongst other things, it is shown that the only coefficients we have to know for the construction of the independent Micu-type Casimir operators of higher degree are those of fourth order for the given Lie superalgebra which are also called graded Lie algebras.

2. Casimir operators of higher degrees of the Backhouse type

One of the methods, with the help of which it is possible to obtain the explicit formulae for the Casimir operators of higher degrees in terms of the independent generators of a given semisimple Lie superalgebra, is that which has been presented by Backhouse (Backhouse 1977). This method represents an immediate generalisation of the Racah method, well known from the classical Lie algebras.

Let the semisimple Lie superalgebra $\mathscr{L}$ be generated by the operators $X_\omega (\omega = 1, 2, \ldots, n)$, for which the relations

$$\langle X_\omega, X_{\omega'}\rangle = C^{\rho}_{\omega\omega'} X_\rho \tag{1}$$

hold.

Here the bracket $\langle\ ,\ \rangle$ is defined as

$$\langle X_\omega, X_{\omega'}\rangle = X_\omega X_{\omega'} - (-1)^{\mathrm{grad}(\omega)\,\mathrm{grad}(\omega')} X_{\omega'} X_\omega, \tag{2}$$

in which $\mathrm{grad}(\omega), \mathrm{grad}(\omega') \in \{0, 1\}$ represent the grades of the corresponding generators X_ω and $X_{\omega'}$; $C^{\rho}_{\omega\omega'}$ is the structure of the corresponding semisimple Lie superalgebra.

The Casimir operators of the nth degree of the Backhouse type are then given by (Scheunert *et al* 1977)

$$K_n = \mathrm{Tr}(\gamma_0 X_{\omega_1} \ldots X_{\omega_n}) X^{\omega_n} \ldots X^{\omega_1}, \tag{3}$$

where $X_{\omega_1}, \ldots, X_{\omega_n}$ are matrices which represent the generators $X_{\omega_1}, \ldots, X_{\omega_n}$ in the specific irreducible representation R of the algebra $\mathscr{L}$. In the irreducible representation the operator γ_0 is realised by a matrix which fulfils the conditions

$$\begin{aligned} &[\gamma_0, X_\omega] = 0 \qquad &&(\text{for } \omega \text{ for which } \mathrm{grad}(\omega) = 0)\\ &\{\gamma_0, X_\omega\} = 0 \qquad &&(\text{for } \omega \text{ for which } \mathrm{grad}(\omega) = 1). \end{aligned} \tag{4}$$

Here, the operators X^ω can be expressed by means of X_ω through the relation

$$X^\omega = g^{\omega}_{\nu} X_\nu, \tag{5}$$

where $g^{\mu\nu}$ are the matrix elements of the inverse matrix of the matrix with elements $g_{\mu\nu} = \mathrm{Tr}(\gamma_0 X_\mu X_\nu)$, whose components realise the metric tensor of the superalgebra $\mathscr{L}$.

The Casimir operators of the nth degree K_n fulfil the equations

$$[K_n, X_n] = 0, \tag{6}$$

i.e., they commute with all the generators of the superalgebra $\mathscr{L}$.

It is evident that if we consider the semisimple Lie algebra L (where all the generators are specified by the zero grade) instead of only the semisimple Lie superalgebra $\mathscr{L}$ we obtain, with the help of the above mentioned construction, the Racah type Casimir operators, provided the irreducible representation R (in which the matrices X_ω are realised) is the adjoint representation of the corresponding algebra L.

For the practical construction of the Casimir operator K_n it is, of course, most advantageous to calculate the coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_1}, \ldots, X_{\omega_n})$ which enter in the Casimir operator K_n in the simplest possible non-trivial representation of the superalgebra $\mathscr{L}$, i.e., in the fundamental lowest-dimensional non-trivial representation of the given algebra.

From the Backhouse construction it is obvious that for the construction of the Casimir operator K_m of the mth degree, it is necessary to know the coefficients of the mth order $\mathrm{Tr}(\gamma_0 X_{\omega_1}, \ldots, X_{\omega_n})$.

Therefore, we need to know for the construction of all independent Casimir operators of the superalgebra osp$(1, 2n)$ the coefficients of all degrees explicitly, i.e., of the second, fourth, ... and $2n$th degree.

However, we can see from the Micu construction (Micu 1964) of the Casimir operators of the Lie algebra of the type Sp$(2n)$ that the only coefficients we have to know are the coefficients of the fourth order. Then, we are able to construct the independent Casimir operators of any higher degree.

An analogous statement remains valid also for the Lie superalgebras osp$(1, 2n)$, for which it is possible by means of the coefficients of the fourth order $\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4})$ to construct the independent operators

$$
\begin{aligned}
T^{(2)}_{\omega_1} &= \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4}) X^{\omega_4} X^{\omega_3} X^{\omega_2}, \\
T^{(3)}_{\omega_1} &= \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4}) T^{(2)\omega_4} X^{\omega_3} X^{\omega_2}, \\
&\vdots \\
T^{(n)}_{\omega_1} &= \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4}) T^{(n-1)\omega_4} X^{\omega_3} X^{\omega_2},
\end{aligned}
\tag{7}
$$

with the help of which the independent Casimir operators of even degree are given by

$$
C_4 = T^{(2)}_{\omega} X^{\omega}, \ldots \qquad C_{2n} = T^{(n)}_{\omega_1} X^{\omega_1}. \tag{8}
$$

The remaining independent Casimir operator of the second degree is simply

$$
C_2 = X^{\omega} X_{\omega}.
$$

The Casimir operators of odd degree such as $C_3, C_5, \ldots$ may be constructed, by means of the third-degree coefficient $\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3})$, in the following way

$$
\begin{aligned}
C_3 &= \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3}) X^{\omega_3} X^{\omega_2} X^{\omega_1}, \\
C_5 &= \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3}) X^{\omega_3} X^{\omega_2} T^{\omega_1}.
\end{aligned}
\tag{9}
$$

A similar method of construction can be used for the higher-degree Casimir operators for any semisimple Lie superalgebra. Of course, for superalgebras of the type osp$(1, 2n)$

$$
\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3}) X^{\omega_3} X^{\omega_2} = k X_{\omega_1} \tag{10}
$$

is valid, where k denotes a numerical normalisation factor. It is obvious that the aforementioned Casimir operators of odd degree C_{2k+1} then reduce themselves to those of even degree C_{2k} and C_{2k+1} do not therefore represent the independent Casimir operators of the superalgebras osp$(1, 2n)$.

3. Casimir operators of the Lie superalgebra osp(1, 4)

In the Racah Weyl basis the algebra osp(1, 4) is defined by the generators X_ω, for which $g(\omega)=0$, which form the underlying Lie subalgebra Sp(4) with the generators E_α, $E_{-\alpha}$ and $H_i(\alpha=1,2,3,4;\ i=1,2)$. To each α there always corresponds the root $\boldsymbol{r}(\alpha)$ (see figure 1).

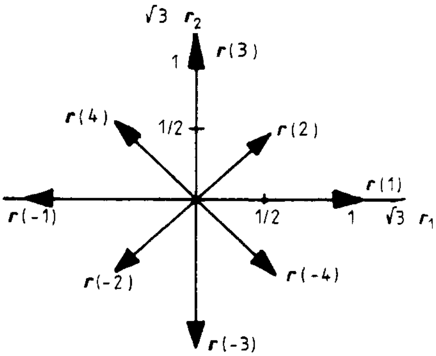

Figure 1. The root diagram of the algebra Sp(4).

The operators H_i are generators of the Cartan subalgebra. The structure of the Sp(4) algebra of operators E_α, H_i is described by the commutation relations (Behrends *et al* 1962) as

$$\begin{aligned} &[H_i, H_j]=0, \qquad [H_i, E_\alpha]=r_i(\alpha)E_\alpha, \\ &[E_\alpha, E_{-\alpha}]=\sum_{i=1}^{2} r_i(\alpha)H_i, \qquad [E_\alpha, E_\beta]=C^\gamma_{\alpha\beta}E_\gamma, \end{aligned} \tag{11}$$

in which the constants $C^\gamma_{\alpha\beta}\neq 0$ only in the case where the equation $\boldsymbol{r}(\gamma)=\boldsymbol{r}(\alpha)+\boldsymbol{r}(\beta)$ is valid. Therefore, if we denote $C^\gamma_{\alpha\beta}$ by $N_{\alpha\beta}=C^\gamma_{\alpha\beta}$, the non-zero structure constants are given by

$$\begin{aligned} N_{24}&=N_{4-2}=N_{-4-2}=N_{2-4}=N_{14}=N_{-21}=N_{-23}=N_{3-4} \\ &=N_{-12}=N_{-4-1}=N_{4-3}=N_{-32}=1/\sqrt{6}. \end{aligned}$$

The other structure constants $r_i(\alpha)$ are obvious from figure 1.

The generators X_ω for which $g(\omega)=1$ holds are denoted by $V_k(k=1,2,3,4)$. Each of these operators is specified in the weight diagram of the fundamental multiplet {4} of the Lie algebra Sp(4) by the corresponding highest weight $\boldsymbol{\lambda}(k)$ (see figure 2).

The commutation and anticommutation relations of the operators H_i, E_α and V_k are given by

$$\begin{aligned} &[E_\alpha, V_k]=\sum_l (e_\alpha)_{lk}V_l, \\ &[H_i, V_k]=\sum_l (h_i)_{lk}V_l, \\ &\{V_kV_{k'}\}=-12\Bigg(\sum_{\alpha=\pm1,\dots,\pm4}(Ce)_{kk'}E_{-\alpha}+\sum_{i=1,2}(Ch_i)_{kk'}H_i\Bigg), \end{aligned} \tag{12}$$

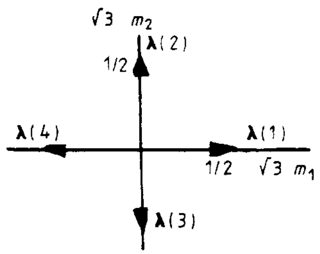

Figure 2. The weight diagram of the fundamental multiplet {4} of the algebra Sp(4).

where $(e_\alpha)_{lk}$ and $(h_i)_{lk}$ are the matrix elements of the four by four matrices which realise the algebra Sp(4) in the fundamental representation and $C_{kk'}$ are the elements of the matrix of 'charge' conjugation (see appendix 1).

Equations (11) and (12) then describe the structure of the Lie superalgebra osp(1, 4) in the Racah Weyl basis.

The fundamental, i.e., the lowest-dimensional representation of the algebra of operators H_i, E_α and V_k is the five-dimensional representation (for its realisation see appendix 1). We are going to apply just this simplest non-trivial representation for the calculation of the coefficients $\mathrm{Tr}(\gamma_0 X_\omega X_{\omega'} \dots)$, with the help of which it is possible to construct the Casimir operators of higher degrees, i.e., in our case the K_2 and K_4.

First of all, we find the components $g_{\omega\omega'}$ of the metric tensor g, defined by the equation

$$g_{\omega\omega'} = \mathrm{Tr}(\gamma_0 X_\omega X_{\omega'}). \tag{13}$$

By using the matrices for X_ω and γ_0 (see appendix 1) we find that the only non-zero components $g_{\omega\omega'}$ of the metric tensor g are

$$g_{\alpha\beta} = \mathrm{Tr}(E_\alpha E_\beta) = \tfrac{1}{6}\delta_{\alpha-\beta} \qquad (\alpha, \beta = \pm 1, \pm 2, \pm 3, \pm 4),$$
$$g_{ij} = \mathrm{Tr}(H_i H_j) = \tfrac{1}{6}\delta_{ij},$$
$$g_{kl} = \mathrm{Tr}(\gamma_0 V_k V_l) = \tfrac{1}{6}C_{kl} \qquad (k, l = 1, 2, 3, 4).$$

The inverse metric tensor g^{-1}, whose components are denoted by $g^{\alpha\beta}$, $((g^{-1})_{\alpha\beta} = g^{\alpha\beta})$, has the following non-zero components

$$g^{\alpha\beta} = 6\delta_{-\alpha\beta} \qquad (\alpha, \beta = \pm 1, \pm 2, \pm 3, \pm 4),$$
$$g^{ij} = 6\delta_{ij} \qquad (i, j = 1, 2),$$
$$g^{kl} = -6C_{kl} \qquad (k, l = 1, 2, 3, 4).$$

If we define the operator X^ω by means of $X_{\omega'}$ by the relation

$$X^\omega = g^{\omega\omega'} X_{\omega'}, \tag{14}$$

then we obtain for the particular components X^ω the relations

$$E^\alpha = 6E_{-\alpha}, \qquad V^2 = -6V_3,$$
$$H^i = 6H_i, \qquad V^3 = 6V_2,$$
$$V^1 = 6V_4, \qquad V^4 = -6V_1.$$

By using these relations, it is possible to write down the quadratic Casimir operator of the superalgebra osp(1, 4)

$$K_2 = \mathrm{Tr}(\gamma_0 X_\omega X_{\omega'}) X^{\omega'} X^\omega = g_{\omega\omega'} X^{\omega'} X^\omega \tag{15}$$

as follows

$$K_2 = 6\left(\sum_{\alpha=1}^{4} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) + H_1^2 + H_2^2 + V_1 V_4 - V_4 V_1 + V_3 V_2 - V_2 V_3\right).$$

If only the even generators E_α, $E_{-\alpha}$ and H_i are retained we obtain the quadratic Casimir operator of the algebra Sp(4).

As

$$\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3}) X^{\omega_3} X^{\omega_2} = \tfrac{5}{4} X_{\omega_1} \tag{16}$$

is valid it is clear that the Casimir operator of the third degree reduces itself to that of the second degree and the following relation holds

$$\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3}) X^{\omega_3} X^{\omega_2} X^{\omega_1} = \tfrac{5}{4} X_{\omega_1} X^{\omega_1}. \tag{17}$$

The Casimir operator of the fourth degree of the superalgebra osp(1, 4) in the Racah Weyl basis can be written in the form

$$K_4 = \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4}) X^{\omega_4} X^{\omega_3} X^{\omega_2} X^{\omega_1}, \tag{18}$$

where the non-zero coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4})$, by which the corresponding terms $X^{\omega_4} X^{\omega_3} X^{\omega_2} X^{\omega_1}$ will contribute to the Casimir operator, are summarised in tables 1 and 2 (see appendix 2).

Using these coefficients the fourth-order Casimir operator can be written down immediately and by using the rules of the superalgebra osp(1, 4), it can be rewritten into the following simpler form

$$K_4 = C_4' + D_4. \tag{19}$$

C_4 is the fourth-order Casimir operator of the underlying algebra Sp(4):

$$\begin{aligned}
\tfrac{1}{9} C_4' = \sum_{\alpha=2,-4} \Bigg[& \sum_{i=1,2} (48 H_i^4 + 2(\{H_i^2, \{E_\alpha, E_{-\alpha}\}\} + \{\{H_i, E_\alpha\}, \{H_i, E_{-\alpha}\}\})) \\
& \times (-1)^{\alpha/2} (\langle H_1 E_\alpha, H_2 E_{-\alpha}\rangle_+^+ + \langle H_1 E_{-\alpha}, H_2 E_\alpha\rangle_+^+) + 4\langle E_{-\alpha} E_\alpha, E_{-\alpha} E_\alpha\rangle_+^+ \\
& + 2(\langle E_\alpha E_{-\alpha}, E_1 E_{-1}\rangle_+^+ + \langle E_\alpha E_{-1}, E_1 E_{-\alpha}\rangle_+^+)\Bigg] \\
& + 2 \sum_{\alpha=2,4} (\langle E_\alpha E_{-\alpha}, E_3 E_{-3}\rangle_+^+ + \langle E_\alpha E_{-3}, E_3 E_{-\alpha}\rangle_+^+) \\
& + 4(\{H_1^2, \{E_1, E_{-1}\}\} + \{H_2^2, \{E_3, E_{-3}\}\}) + 2\{E_1, E_{-1}\} + 2\{E_3, E_{-3}\} \\
& + 8 \sum_{\beta=1,3} \langle E_\beta E_{-\beta}, E_\beta E_{-\beta}\rangle_+^+ + \frac{1}{6\sqrt{3}} (H_2 - H_1) \qquad (20) \\
& + 4(\langle E_2 E_{-3}, E_2 E_{-1}\rangle_+^+ + \langle E_{-2} E_3, E_{-2} E_1\rangle_+^+ - \langle E_4 E_{-3}, E_4 E_1\rangle_+^+ \\
& - \langle E_{-4} E_3, E_{-4} E_{-1}\rangle_+^+) - \sqrt{2}(\langle E_{-2} E_3, E_{-4} H_1\rangle_+^+ + \langle E_{-2} H_1, E_{-4} E_3\rangle_+^+ \\
& + \langle E_2 E_{-3}, E_4 H_1\rangle_+^+ + \langle E_2 H_1, E_4 E_{-3}\rangle_+^+ + \langle E_{-2} E_1, E_4 H_2\rangle_+^+
\end{aligned}$$

$$+\langle E_{-2}H_2, E_4E_1\rangle_+^+ +\langle E_2E_{-1}, E_{-4}H_2\rangle_+^+ +\langle E_2H_2, E_{-4}E_{-1}\rangle_+^+)$$
$$+\sqrt{2}(\langle H_2E_2, E_4E_{-3}\rangle_+^+ -\langle H_2E_4, E_2E_{-3}\rangle_+^+ +\langle H_1E_{-2}, E_4E_1\rangle_+^+$$
$$-\langle H_1E_4, E_{-2}E_1\rangle_+^+ +\langle H_2E_{-2}, E_{-4}E_3\rangle_+^+ -\langle H_2E_{-4}, E_{-2}E_3\rangle_+^+$$
$$+\langle H_1E_2, E_{-4}E_{-1}\rangle_+^+ -\langle H_1E_{-4}, E_2E_{-1}\rangle_+^+)$$

and the part D_4 with the odd operators V_α (see appendix 2) is given by

$$\begin{aligned}\tfrac{1}{9}D_4 = &-\langle E_2H_1, V_4V_3\rangle_+^- -\langle H_1V_4, V_3E_2\rangle_-^- +\langle H_1E_4, V_3V_1\rangle_+^- \\
&+\langle E_4V_3, V_1H_1\rangle_-^- +\langle H_2E_2, V_4V_3\rangle_+^- +\langle E_2V_4, V_3H_2\rangle_-^- \\
&+\langle E_4H_2, V_3V_1\rangle_+^- +\langle H_2V_3, V_1E_4\rangle_-^- -\langle H_2E_{-2}, V_1V_2\rangle_+^- \\
&-\langle E_{-2}V_1, V_2H_2\rangle_-^- +\langle E_{-4}H_2, V_2V_4\rangle_+^- +\langle H_2V_2, V_4E_{-4}\rangle_-^- \\
&+\langle H_1E_{-4}, V_2V_4\rangle_+^- +\langle E_{-4}V_2, V_4H_1\rangle_-^- +\langle E_{-2}H_1, V_1V_2\rangle_+^- \\
&+\langle H_1V_1, V_2E_{-2}\rangle_-^- -\langle H_1^2, V_4V_1\rangle_+^- -\langle V_4H_1, H_1V_1\rangle_-^- \\
&+\langle H_2^2, V_3V_2\rangle_+^- +\langle V_3H_2, H_2V_2\rangle_-^- +\langle E_{-2}E_2, V_3V_2\rangle_+^- \\
&+\langle E_2V_3, V_2E_{-2}\rangle_-^- +\langle E_{-4}E_4, V_3V_2\rangle_+^- +\langle E_4V_3, V_2E_{-4}\rangle_-^- \\
&-\langle E_4E_{-4}, V_4V_1\rangle_+^- -\langle E_{-4}V_4, V_1E_4\rangle_-^- \\
&+2(\langle E_{-3}E_3, V_3V_2\rangle_+^- +\langle E_3V_3, V_2E_{-3}\rangle_-^- -\langle E_{-1}E_1, V_4V_1\rangle_+^- \\
&-\langle E_1V_4, V_1E_{-1}\rangle_-^-) +\langle V_1V_2, V_3V_4\rangle_+^+ -\langle V_1V_3, V_2V_4\rangle_+^+ \\
&+\{[V_1, V_4], [V_2, V_3]\} +4(\{V_1V_4, V_4V_1\} \\
&-\{V_1V_1, V_4V_4\} +\{V_2V_3, V_3V_2\} -\{V_2V_2, V_3V_3\}) \\
&+(4/\sqrt{6})(\{E_1, V_4V_4\} -\{E_{-1}, V_1V_1\} +\{E_3, V_3V_3\} \\
&-\{E_{-3}, V_2V_2\}) -2\sqrt{2}(\langle H_1V_4, V_4E_1\rangle_-^- +\langle H_1V_1, V_1E_{-1}\rangle_-^- \\
&+\langle H_2V_3, V_3E_3\rangle_-^- +\langle H_2V_2, V_2E_{-3}\rangle_-^-)\end{aligned} \tag{21}$$

where the brackets $\langle AB, CD\rangle_+^\pm$ and $\langle AB, CD\rangle_-^\pm$ are defined with the help of commutators [,] or anticommutators { , } as follows:

$$\begin{aligned}\langle AB, CD\rangle_+^\pm &= \{AB, CD\} \pm \{DC, AB\},\\ \langle AB, CD\rangle_-^\pm &= [AB, CD] \pm [DC, BA].\end{aligned} \tag{22}$$

Their symmetric properties with respect to the corresponding permutations, e.g.

$$\langle AB, CD\rangle_+^+ = \langle DC, BA\rangle_+^+ = \langle CD, AB\rangle_+^+ = \langle BA, DC\rangle_+^+ \text{ etc}, \tag{23}$$

are obvious.

Let us turn now to the construction of the Casimir operators of higher degrees of the superalgebra osp$(1, 2n)$ and to the role of the fourth-order coefficients in them.

The Backhouse type of Casimir operator of the sixth degree of the superalgebra osp$(1, 6)$ will be of the form

$$K_6 = \mathrm{Tr}(\gamma_0 X_{\omega_1} \dots X_{\omega_6}) X^{\omega_6} \dots X^{\omega_1}. \tag{24}$$

According to relations (7) and (8) the Casimir operators of the sixth degree can be written down in the form

$$C_6 = T^{(3)}_{\omega_1} X^{\omega_1}, \tag{25}$$

where

$$T^{(3)}_{\omega_1} = \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4}) T^{(2)\omega_4} X^{\omega_3} X^{\omega_2}$$

and

$$T^{(2)\omega_4} = g^{\omega_4\omega_3} T^{(2)}_{\omega_3}$$
$$= g^{\omega_4\omega_3}\,\mathrm{Tr}(\gamma_0 X_{\omega_3} X_{\omega_4} X_{\omega_5} X_{\omega_6}) X^{\omega_6} X^{\omega_5} X^{\omega_4}.$$

Here the metric tensor $g^{\omega_4\omega_3}$ is defined and calculated for the superalgebra osp$(1, 2n)$, $n = 2$ in relations (13). As a result we obtain the Casimir operator C_6 of the sixth degree of the superalgebra osp(1, 6) in the form

$$C_6 = \mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4})\,\mathrm{Tr}(\gamma_0 X_{\omega_4} X_{\omega_3})$$
$$\times \mathrm{Tr}(\gamma_0 X_{\omega_3} X_{\omega_4} X_{\omega_5} X_{\omega_6}) X^{\omega_6} X^{\omega_5} X^{\omega_4} X^{\omega_3} X^{\omega_2} X^{\omega_1}. \tag{26}$$

The simplest way of calculating the coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4})$ is their calculation in the fundamental representation, which is in the given case of the superalgebra osp(1, 6) of dimension seven.

If the generators of the underlying Lie superalgebra Sp(6) are represented by matrices for the indices $i, j = 1, \ldots, 6$ the matrix γ_0 is represented by a diagonal matrix $\gamma_0 = (\gamma_{0ij})$ where $\gamma_{0ij} = \delta_{ij}$ for the indices $i, j = 1, \ldots, 6$ and $\gamma_{77} = -1$. The odd operators of the considered Lie superalgebra are then represented by seven by seven matrices with non-zero matrix elements in the seventh row and seventh column only. Their explicit form can be obtained by a similar procedure to that which has been used for the case of the superalgebra osp(1, 4) (see the first part of § 3 and appendix 1).

With this basis of our knowledge of the fundamental representation it is possible to calculate the coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_{i_1}} \ldots X_{\omega_{i_4}})$ and with the aid of them to construct all the dependent invariants of the superalgebra osp(1, 6) i.e. the higher invariants of the fourth and sixth degree.

4. Conclusion

We have found the explicit formulae for the Casimir operators of the second (equation (12)) and fourth degree (equations (13) and (16)), in terms of the generators E_α, H_i and V_α for the graded algebra osp(1, 4) and its underlying algebra Sp(4).

In the same basis, Micu in his paper suggests to construct the Casimir operators of the Lie algebra Sp(4) not with the help of the above mentioned coefficients $\mathrm{Tr}(X_\omega X_\rho X_\lambda X_\delta)$, but by means of the symmetrised coefficients $[\omega\rho\lambda\delta]$ defined through the traces $\mathrm{Tr}(X_\omega X_\rho X_\lambda X_\delta)$ by the relation

$$[\omega\rho\lambda\delta] = \tfrac{1}{6}\,\mathrm{Tr}(X_\omega X_\rho X_\lambda X_\delta + X_\omega X_\lambda X_\rho X_\delta + X_\rho X_\omega X_\lambda X_\delta$$
$$+ X_\rho X_\lambda X_\omega X_\delta + X_\lambda X_\omega X_\rho X_\delta + X_\lambda X_\rho X_\omega X_\delta). \tag{27}$$

It is evident that the coefficients $[\omega\rho\lambda\delta]$ are totally symmetric in all four indices. The

Casimir operator of the Micu type for the algebra Sp(4) is then given by

$$\sum_{\omega\rho\lambda\delta} [\omega\rho\lambda\delta] X^{\delta} X^{\lambda} X^{\rho} X^{\omega}. \tag{28}$$

The only non-zero coefficients in this sum are

$$\begin{array}{lll}
[e_1e_{-1}h_1h_1]=\frac{1}{216}, & [h_1h_1h_1h_1]=\frac{1}{72}, & [h_1h_1e_4e_{-4}]=\frac{1}{216}, \\
[e_1e_{-2}e_4h_2]=-\frac{1}{216}2^{-1/2}, & [h_2h_2h_2h_2]=\frac{1}{72}, & [h_2h_2e_{-4}e_4]=\frac{1}{216}, \\
[e_2e_{-2}h_1h_2]=\frac{1}{216}, & [h_2h_2e_3e_{-3}]=\frac{1}{216}, & [h_1h_2e_2e_{-2}]=-\frac{1}{432}, \\
[e_1e_{-1}e_1e_{-1}]=\frac{1}{108}, & [h_2h_2e_2e_{-2}]=\frac{1}{216}, & [h_1h_2e_4e_{-4}]=\frac{1}{432}, \\
[e_2e_2e_{-2}e_{-2}]=\frac{1}{216}, & [e_1e_3e_{-2}e_{-2}]=\frac{1}{216}, & [h_1e_{-3}e_2e_4]=-\frac{1}{216}2^{-1/2}, \\
[e_4e_4e_{-4}e_{-4}]=\frac{1}{216}, & [e_{-1}e_3e_{-4}e_{-4}]=-\frac{1}{216}, & [e_3e_3e_{-3}e_{-3}]=\frac{1}{108}, \\
[e_2e_{-4}e_{-2}e_4]=\frac{1}{216}, & & \\
[e_3e_{-3}e_2e_{-2}]=\frac{1}{216}, & [e_3e_{-3}e_4e_{-4}]=\frac{1}{216}, & [e_1e_{-1}e_4e_{-4}]=\frac{1}{216}, \\
[e_1e_{-1}e_2e_{-2}]=\frac{1}{216}, & &
\end{array} \tag{29}$$

and the terms which arise from the above mentioned by the change of the root $\boldsymbol{r}$ to $-\boldsymbol{r}$ at all the generators which enter in the corresponding product. Apart from the last four brackets [, , ,] all the others are given in the paper by Micu (1964) (the last four brackets were obviously forgotten by him).

The Casimir operator of the Micu type is formally evidently much simpler than the Casimir operator

$$\sum \operatorname{Tr}(X_{\omega} X_{\lambda} X_{\rho} X_{\delta}) X^{\delta} X^{\rho} X^{\lambda} X^{\omega}. \tag{30}$$

The open problem which is left is, if a possibility of defining a generalisation of the brackets for the whole graded Lie algebra osp(1, 4) exists, i.e. whether a possibility of finding the formally simpler formula for the Casimir operator of the fourth degree even for the case of the graded algebra exists.

Appendix 1.

The explicit fundamental (five-dimensional) representation of the graded algebra osp(1, 4) (Bednář and Šachl 1978b)

$$E_1=\frac{1}{\sqrt{6}}\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad E_{-1}=\frac{1}{\sqrt{6}}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$E_2=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad E_{-2}=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$
\begin{aligned}
&E_3=\frac{1}{\sqrt{6}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \qquad
E_{-3}=\frac{1}{\sqrt{6}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix},\\
&E_4=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \qquad
E_{-4}=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix},\\
&H_1=\frac{1}{2\sqrt{3}}\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \qquad
H_2=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix},\\
&V_1=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot \end{pmatrix}, \qquad
V_2=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&-1&\cdot&\cdot \end{pmatrix},\\
&V_3=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \qquad
V_4=\frac{1}{2\sqrt{3}}\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1\\ -1&\cdot&\cdot&\cdot&\cdot \end{pmatrix},\\
&\gamma_0=\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&-1 \end{pmatrix}, \qquad
C=\begin{pmatrix} \cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&+1&\cdot&\cdot\\ \cdot&-1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}.
\end{aligned}
\tag{A1}
$$

Appendix 2.

The coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_1} X_{\omega_2} X_{\omega_3} X_{\omega_4})$ of the Casimir operator of the fourth degree of the superalgebra osp(1, 4) with all even generators and those which contain the odd generators can be given in the form of tables 1 and 2 respectively. The corresponding permutations of these generators are given in the first column.

Table 1. The coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_1}, X_{\omega_2}, X_{\omega_3}, X_{\omega_4})$ of the GLA osp(1, 4) and LA Sp(4) with even generators X_ω only.

Numerical factor	$\frac{1}{144}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{144}$	$\frac{1}{144}$
Basic permut. / Possible perm. order of generators	$H_1H_1H_1H_1$	$H_1H_1E_1E_{-1}$	$H_2H_2E_3E_{-3}$	$H_2H_2H_2H_2$	$H_1H_1E_2E_{-2}$	$H_2H_2E_2E_{-2}$	$H_1H_1E_{-4}E_4$	$H_2H_2E_{-4}E_4$	$H_1H_2E_2E_{-2}$	$H_1H_2E_{-4}E_4$	$H_1E_{-3}E_2E_4$	$H_2E_{-1}E_2E_{-4}$	$E_1E_1E_{-1}E_{-1}$	$E_3E_3E_{-3}E_{-3}$	$E_2E_2E_{-2}E_{-2}$	$E_{-4}E_{-4}E_4E_4$
1 2 3 4	2	1	1	2	1	1	1	1								
1 2 4 3	2	1	1	2	1	1	1	1								
1 3 2 4	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
1 3 4 2	2	1	1	2	1	1	1	1								
1 4 2 3	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
1 4 3 2	2	1	1	2	1	1	1	1								
2 1 3 4	2	1	1	2	1	1	1	1								
2 1 4 3	2	1	1	2	1	1	1	1								
2 3 1 4	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
2 3 4 1	2	1	1	2	1	1	1	1								
2 4 1 3	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
2 4 3 1	2	1	1	2	1	1	1	1								
3 1 2 4	2	1	1	2	1	1	1	1								
3 1 4 2	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
3 2 1 4	2	1	1	2	1	1	1	1								
3 2 4 1	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
3 4 1 2	2	1	1	2	1	1	1	1								
3 4 2 1	2	1	1	2	1	1	1	1								
4 1 2 3	2	1	1	2	1	1	1	1								
4 1 3 2	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
4 2 1 3	2	1	1	2	1	1	1	1								
4 2 3 1	2	−1	−1	2					−1	1	−1	−1	1	1	2	2
4 3 1 2	2	1	1	2	1	1	1	1								
4 3 2 1	2	1	1	2	1	1	1	1								

Table 1. (continued)

Numerical factor	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{144}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$
Possible perm. order of generators \ Basic permut.	$E_1E_3E_{-2}E_{-2}$	$E_{-1}E_3E_{-4}E_{-4}$	$E_2E_{-4}E_{-2}E_4$	$E_3E_{-4}E_{-2}H_2$	$E_{-1}E_{-4}E_2H_1$	$E_3E_{-3}E_2E_{-2}$	$E_3E_{-3}E_{-4}E_4$	$E_1E_{-1}E_{-4}E_4$	$E_1E_{-1}E_2E_{-2}$	$H_1E_3E_{-2}E_{-4}$	$H_2E_1E_{-2}E_4$	$E_{-1}E_{-3}E_2E_2$	$E_1E_{-3}E_4E_4$	$E_{-3}E_4E_2H_2$	$E_1E_4E_{-2}H_1$
1 2 3 4			−1	1	1	1		1	1					1	1
1 2 4 3			1				1								
1 3 2 4	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
1 3 4 2			1				1								
1 4 2 3	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
1 4 3 2			−1	1	1	1		1	1					1	1
2 1 3 4			1				1								
2 1 4 3			−1	1	1	1		1	1					1	1
2 3 1 4	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
2 3 4 1			−1	1	1	1		1	1					1	1
2 4 1 3	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
2 4 3 1			1				1								
3 1 2 4			1				1								
3 1 4 2	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
3 2 1 4			−1	1	1	1		1	1					1	1
3 2 4 1	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
3 4 1 2			−1	1	1	1		1	1					1	1
3 4 2 1			1				1								
4 1 2 3			−1	1	1	1		1	1					1	1
4 1 3 2	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
4 2 1 3			1				1								
4 2 3 1	1	−1	1	−1	−1					−1	−1	1	−1	−1	−1
4 3 1 2			1				1								
4 3 2 1			−1	1	1	1		1	1					1	1

Table 2. The coefficients $\mathrm{Tr}(\gamma_0 X_{\omega_1}, X_{\omega_2}, X_{\omega_3}, X_{\omega_4})$ of the GLA osp(1, 4) which contain the odd operators V_α.

Numerical factor	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{72\sqrt{2}}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{72}$	$\frac{1}{144}$	$\frac{1}{72}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$
Basic permut. / Possible perm. order of generators	$V_1V_4H_1H_1$	$V_1V_1V_4V_4$	$V_2V_3H_2H_2$	$V_2V_2V_3V_3$	$V_1V_1E_{-1}H_1$	$V_4V_4E_1H_1$	$V_2V_2E_{-3}H_2$	$V_3V_3E_3H_2$	$V_2V_4E_{-4}H_1$	$V_1V_2E_{-2}H_2$	$V_1V_4E_4E_{-4}$	$V_3V_1E_4H_1$	$V_4V_3E_2H_2$	$V_1V_2E_{-2}H_1$	$V_1V_4E_1E_{-1}$	$V_2V_4E_{-4}H_2$	$V_2V_3E_3E_{-3}$	$V_3V_1E_4H_2$	$V_2V_3E_2E_{-2}$	$V_2V_3E_4E_{-4}$	$V_4V_3E_2H_1$	$V_1V_2V_3V_4$
1 2 3 4	−1	−1	1	−1	1	1	1	1						−1	−1	−1	1	−1	1	1	1	1
1 2 4 3	−1	−1	1	−1	−1	−1	−1	−1	−1	1	−1	−1	−1									
1 3 2 4																						−1
1 3 4 2	−1	1	1	1	−1	−1	−1	−1	−1	1	−1	−1	−1									
1 4 2 3																						1
1 4 3 2	−1	1	1	1	1	1	1	1						−1	−1	−1	1	−1	1	1	1	−1
2 1 3 4	1	−1	−1	−1	1	1	1	1	1	−1	1	1	1									
2 1 4 3	1	−1	−1	−1	−1	−1	−1	−1						1	1	1	−1	1	−1	−1	−1	1
2 3 1 4																						1
2 3 4 1	1	1	−1	1	−1	−1	−1	−1						1	1	1	−1	1	−1	−1	−1	−1
2 4 1 3																						−1
2 4 3 1	1	1	−1	1	1	1	1	1	1	−1	1	1	1									
3 1 2 4	−1	1	1	1	−1	−1	−1	−1	−1	1	−1	−1	−1									
3 1 4 2																						−1
3 2 1 4	1	1	−1	1	−1	−1	−1	−1						1	1	1	−1	1	−1	−1	−1	−1
3 2 4 1																						1
3 4 1 2	−1	−1	1	−1	1	1	1	1						−1	−1	−1	1	−1	1	1	1	1
3 4 2 1	1	−1	−1	−1	1	1	1	1	1	−1	1	1	1									
4 1 2 3	−1	1	1	1	1	1	1	1						−1	−1	−1	1	−1	1	1	1	−1
4 1 3 2																						1
4 2 1 3	1	1	−1	1	1	1	1	1	1	−1	1	1	1									
4 2 3 1																						−1
4 3 1 2	−1	−1	1	−1	−1	−1	−1	−1	−1	1	−1	−1	−1									
4 3 2 1	1	−1	−1	−1	−1	−1	−1	−1						1	1	1	−1	1	−1	−1	−1	1

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